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Elliott's World: From Square Ice to Cubic Jellium[1]

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Happy Birthday, Elliott! This is an after-dinner talk and not a scientific communication, so I will take the opportunity to reminisce about memorable moments in Elliott's life. I don't pretend to cover all aspects of Elliott's world. The title of the talk should not be "Elliott's World" but rather "The Little Bits of Elliott's World that I have Shared." I have been a friend of Elliott for about forty years, but our interactions have been sporadic rather than continuous. His path and mine have been like one-dimensional Brownian motions, with points of intersection infinite in number but nowhere dense in time. Luckily, in the early days before we all became addicted to e-mail, he wrote me some hand-written letters which I still treasure. Instead of making up stories based on my unreliable memory, I can give you direct quotes from his letters.

I don't remember where and when we first met. It was probably at one of Joel Lebowitz's Statistical Mechanics meetings in the early years when the meetings were at Yeshiva University. In those early years, Elliott and I were both enjoying ourselves solving physics problems in one dimension. We shared a taste for exact analytic solutions. We preferred to play around in a one-dimensional world where problems could be solved exactly, instead of struggling with the real world where exact solutions hardly ever exist. In 1966 Elliott and Daniel Mattis published their splendid book, "Mathematical Physics in One Dimension," summarizing what we knew about the one-dimensional universe.(4)

Then at the beginning of 1967 came a big surprise. Elliott moved up from one dimension to two. He published a short paper in Physical Review Letters with an exact solution of the two-dimensional ice problem.(5) Two-dimensional ice means a square lattice with an oxygen atom at every vertex

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and a hydrogen atom on every bond attached to one of the adjacent oxygens, with the rule that two and only two hydrogens are adjacent to each oxygen. He proved that the number of configurations of square ice consistent with this rule is asymptotically equal to W to the power N, where N is the number of oxygens and W is four-thirds to the power three-halves. This magic number W, four-thirds to the power three-halves, comes out of a delicate piece of analysis invented for another purpose by Frank Yang. I recognized at once that Elliott had moved into a new league where I could no longer compete. I wrote to congratulate him and here is what he wrote back.

"Let me make some comments. (a) I agree that many people will try to solve this by another method, but I doubt very much that they will succeed… (b) The ice solution is only a curtain-opener. I can also solve the two-dimensional F model of an antiferroelectric. It has a phase transition with unusual properties, a natural boundary in the complex temperature plane. Also, unlike the Ising model, it can be solved in an arbitrary external field. Again it is unusual: below the transition temperature there is no polarization unless the external field exceeds a critical value. At the moment I am writing this up…" That letter gives you a good picture of Elliott in action, not just solving a hard mathematical puzzle but looking beyond the mathematics to understand the physics. I could do the mathematics as well as he could, but I could not match his physical intuition.

A few months later I received a letter from Elliott on an entirely different subject, the war in Vietnam. We agreed that the war was stupid and evil, but we disagreed about its importance. I said it was only a typical colonial war, similar to the colonial wars that had been fought in the recent past, by Britain in Palestine and by France in Vietnam. I thought the United States would soon get tired of it and give up the struggle and then life would continue as before. Elliott disagreed because he saw the war as a moral issue and not merely a political issue. He sent me a manifesto opposing the war on moral grounds and invited me to sign it. I did not sign it and he wrote me a letter which I treasure as a statement of his moral commitment. Since our government is now threatening to plunge us into another war that raises similar moral concerns, I am glad to be able to read you a piece of his letter. He is writing in July 1967, with the war still escalating and no end in sight.

"I ask myself how I would feel if I were born fifteen years later… I would feel my chances of being drafted very high, and being fifteen years younger I would consider involvement in the war to be at least as odious as I do now with fifteen additional years in which to become hardened. I would assume that being drafted meant a fair probability of being involved in the actual fighting, and this in turn would mean a possibility of being ordered to burn a village, bomb a hospital or something equally disthese two titans against the dark forces of quantum statistics. After heroic efforts, we hammered together a long and complicated proof. We sent off a short paper to Physical Review Letters claiming victory. But then Jürg Fröhlich looked carefully at our proof and found a hole in it. An inequality that we had proved for real matrices is not true for complex matrices. As a result, our proof[7] works for antiferromagnets but not for ferromagnets. Physical intuition and experimental evidence tell us that long-range order should exist in ferromagnets and antiferromagnets alike. The hole in our proof shows that it failed to capture the essential physics of the problem. There must be a better way, but we did not find it.

Meanwhile, Elliott was engaged in a far more fruitful collaboration with Walter Thirring, finding a simple and physically well-motivated proof of the stability of matter. Andrew Lenard and I had found a proof of the stability of matter in 1967. Our proof was so complicated and so unilluminating that it stimulated Elliott and Walter to find the first decent proof.[6] Why was our proof so bad and why was theirs so good? The reason is simple. Lenard and I began with mathematical tricks and hacked our way through a forest of inequalities without any physical understanding. Elliott and Walter began with a physical idea, that matter in bulk could be well approximated by the Thomas–Fermi model which was originally designed to be a model for a single heavy atom. They then went on to find the appropriate mathematical language to make their physical understanding rigorous. They studied the Thomas–Fermi model in detail and understood the mathematical reasons why it does a good job of describing many-Fermion systems. Once they had the right mathematical language, they could finish the proof of stability in a couple of pages. Our proof was a dead end. Theirs was a gateway to a new world of ideas.

For twenty-five years since Elliott came to Princeton, he has continued to work with a succession of students and colleagues, many of them here today, to deepen our understanding of atomic physics. Some people say, once you have written down the Schrödinger equation for an atom, you have a final theory of atomic physics and nothing interesting remains to be done. Elliott says, on the contrary, when you look at the behavior of atoms carefully and imaginatively, surprises and mysteries are still lurking in every corner. One example of a surprise that Elliott uncovered is the behavior of a heavy atom in a strong magnetic field, when the number Z of electrons and the strength B of the field are both large. Jakob Yngvason[10] had shown that there is a Thomas–Fermi theory which is asymptotically exact when Z tends to infinity while B is held fixed, contrary to what Elliott and many others had thought. Stimulated by Yngvason's work to explore the problem further, Elliott found that the atom behaves in five qualitatively different ways in five different ranges of values of Z and B. These complex patterns of behavior may actually occur in the crusts of neutron stars, where neutron-rich atoms with high Z coexist with magnetic fields of the order of ten to the thirteenth gauss. Atoms in such an environment may have shapes and sizes quite different from those with which we are familiar.[8]

Elliott has always liked to explore situations where many-particle systems show surprising behavior because of subtle effects of quantum mechanics. Another famous example of such a situation is the jellium problem. Jellium is a mythical substance composed of positively charged boson particles immersed in a uniform negatively charged background. Leslie Foldy discovered long ago,[3] that the binding energy of each particle in jellium is a constant times the fourth root of the density. Foldy's calculation made no claim to be rigorous. Elliott attacked the problem with his usual rigor, chopping the jellium into cubical boxes so that it looked like a cubist painting. In 1988 he was able to prove that the binding energy goes with the fourth root of the density, but he was not able to verify the numerical coefficient in Foldy's formula.[2] Finally, two years ago, a triumphant e-mail message came from Elliott: "You might be interested in the recent paper that shows that Foldy's 1961 calculation for jellium, following Bogolubov's method, is exactly right." After forty years, Elliott and Jan Solovej had nailed down the ground-state energy of jellium.[9]

When Elliott spoke at the Northwestern symposium in 1969, he said: "The ideal after-dinner speaker should aid the digestive juices by being witty; I am not one of those. At physics dinners, serious speakers can be boring, not for the reason that they have nothing to say, but usually because they will not say what is on their minds and close to their hearts. I do not intend to fall into that category either." Elliott spoke from the heart then and he still speaks from the heart today. He is a worthy successor to Wolfgang Pauli. Like Pauli, he has earned the title, "the conscience of physics," by always speaking his mind and never hesitating to call nonsense nonsense.

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